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On 2D Bisection Method for Double Eigenvalue Problems

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The two-dimensional bisection method presented in (SIAM J. Matrix Anal. Appl. 13(4), 1085 (1992)) is efficient for solving a class of double eigenvalue problems. This paper further extends the 2D bisection method to full matrix cases and analyses its stablity. As in a single parameter case, the 2D bisection method is very stable for the tridiagonal matrix triples satisfying the symmetric-definite condition. Since the double eigenvalue problems arise from two-parameter boundary value problems, an estimate of the discretization error in eigenpairs is also given. Some numerical examples are included.

1. INTRODUCTION

This paper considers the numerical solution of the following double eigenvalue problems

$$\begin{aligned}(A_0 - \lambda A_1 - \mu A_2)\, y_1 &= 0,\\ (B_0 - \lambda B_1 - \mu B_2)\, y_2 &= 0,\end{aligned} \tag{1.1}$$

where the real pairs (λ, μ) and the nonzero tensor products $y_1 \otimes y_2$, $y_1 \in R^n$, $y_2 \in R^m$, are the eigenvalues and the corresponding eigenvectors to be found. In [24] the author presented a two-dimensional bisection method for the above problems. To apply this method, however, the coefficient matrices have to satisfy the so-called "TBC" condition: $A_0 \in R^{n\times n}$, $B_0 \in R^{m\times m}$ are irreducible symmetric tridiagonal matrices; $A_i \in R^{n\times n}$, $B_i \in R^{m\times m}$ $(i = 1, 2)$ are nonsingular diagonal matrices with the diagonal entries of the same sign, respectively. This paper is to extend this method to full matrix cases, i.e., $A_i \in R^{n\times n}$, $B_i \in R^{m\times m}$ are symmetric for $i = 0, 1, 2$ and positive or negative definite for $i = 1, 2$ (symmetric-definite condition for short). The stability of the 2D bisection is analyzed. In addition, the bound of the discretization error is also estimated since the problems (1.1) arise by some discrete technique from the differential eigenvalue problems as

$$\begin{aligned}
&-(p_1(x_1)\, y_1')' + q_1(x_1)\, y_1 = (\lambda s_{11}(x_1) + \mu s_{12}(x_1))\, y_1,\\
&-(p_2(x_2)\, y_2')' + q_2(x_2)\, y_2 = (\lambda s_{21}(x_2) + \mu s_{22}(x_2))\, y_2,\\
&y_1(a_1)\cos\alpha_1 - (p_1 y_1')(a_1)\sin\alpha_1 = 0,\\
&y_1(b_1)\cos\beta_1 - (p_1 y_1')(b_1)\sin\beta_1 = 0,\\
&y_2(a_2)\cos\alpha_2 - (p_2 y_2')(a_2)\sin\alpha_2 = 0,\\
&y_2(b_2)\cos\beta_2 - (p_2 y_2')(b_2)\sin\beta_2 = 0,
\end{aligned} \tag{1.2}$$

where $0 \le \alpha_i < \pi$, $0 < \beta_i \le \pi$, $p_i > 0$ and p_i', q_i, s_{ij} are real valued and continuous, for $i, j = 1, 2$. The systems (1.2) are known as two-parameter Sturm–Liouville (S-L) eigenvalue problems which arise in many practical applications related to mathematical physics and engineering problems. For instance, the charge-singularity problem [28] from the electromagnetic field (also cf. Section 5), the self-consistent field (SCF) equation in proton dynamics,

$$\begin{aligned}\{T_\xi - \varepsilon_{\mathrm{SCF}}\varphi_1(\xi, \eta) - \lambda_\eta\}X(\xi) &= 0,\\ \{T_\eta - \varepsilon_{\mathrm{SCF}}\varphi_2(\xi, \eta) - \lambda_\xi\}Y(\eta) &= 0,\end{aligned}$$

where T_ξ, T_η are second-order differential operators, $\varepsilon_{\mathrm{SCF}}$ and $X(\xi)$, $Y(\eta)$ are the common energy eigenvalue and eigenfunctions, and the coupling constants $(\lambda_\eta, \lambda_\xi)$ are connected by some formulae; cf. [15] for details. In subsonic aerodynamics, the "delta wing problem" also gives rise to the problem (1.2). A delta wing is idealized as an infinite sector. The delta wing problem is to determine the nature of the solution of the potential equation in the neighborhood of the tip of the wing. The governing equations for flows over a delta wing which only weakly disturb the oncoming uniform flows can be reduced to the equations (cf. [38])

$$\begin{aligned}
A''(\alpha) + \{h - \nu(\nu+1)k^2 sn^2(\alpha, k)\}A(\alpha) &= 0,\\
A(K) = A(-K) &= 0;\\
B''(\beta) + \{-h + \nu(\nu+1)(1 - k'^2 sn^2(\beta, k'))\}B(\beta) &= 0,\\
B(-K') = B'(K') &= 0,
\end{aligned}$$

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where h and $\nu(\nu + 1)$ are the spectral parameters, $k = \sin \omega$, $k' = \cos \omega$, ω is the half angle of the sector, and $sn(\cdot)$ is the Jacobian elliptic functions.

Probably the most natural way in which two-parameter S-L problems arise in practice is when the technique of separation of variables is applied to certain boundary value problems associated with partial differential equations [13, 29, 39]. The Mathieu's, Lamé's, and ellipsoidal wave equations are famous examples [1, 2, 36]. Two-parameter S-L problems also arise in a number of other areas; for instance, in the theory of approximations, in many-body diffraction theory, and in nonlinear control problems (cf. [34]).

Two-parameter eigenvalue problems have been thoroughly studied in theory (cf. [3, 6, 16]); the work of the corresponding numerical analysis has also attracted a lot of attention (e.g., cf. [10, 12, 18, 21–23, 30]). In particular, [5, 9] had successfully extended the Prüfer method, which is one of the most efficient and powerful techniques for ordinary S-L eigenvalue problems [25, 27], to two-parameter cases. Since the Prüfer method is a kind of shooting method, a good starting value is a key to success. Although both [5, 9] gave the strategy to produce starting values, the problem cannot be considered completely solved because the two-parameter situation is much more complicated than a single parameter case. Matrix-type methods, like 2D bisection, provide another approach. Not only can they produce starting values for the 2D Prüfer method, but they also have the advantage of simplicity and maintain their efficiency even for high oscillation solutions (cf. [25]). Therefore, it is necessary to develop matrix-type methods for two-parameter eigenvalue problems.

This paper mainly concerns computation and related theoretical analysis of the eigenpairs. It does not mean that computation of the eigenvectors is a trivial task. Since the numerical solution of three-dimensional partial differential equations often leads to very large systems of equations and poses serious complexity problems [14, 31], it seems worthwhile to consider the classical method of separation of variables for elliptic boundary-value problems [11, 21]. In doing this, the computation of eigenvectors plays a major role. The importance of the eigenvector's computation cannot be overemphasized and will be treated as the subject of another paper [26].

The format of this paper is organized in the following way. The concept of the eigencurve and the extension of the 2D bisection method are described in Section 2. Section 3 presents stability analysis for the 2D bisection by Wilkinson's backward error analysis technique. Section 4 discusses the definiteness condition and discretization error of double matric eigenproblems from double differential eigenproblems. Numerical examples are given in Section 5.

2. EIGENCURVES AND TWO-DIMENSIONAL BISECTION

In order to decribe the 2D bisection method, we introduce the concept of eigencurves of double eigenvalue problems. Let

$$\begin{aligned} f(\lambda, \mu) &= \det(A_0 - \lambda A_1 - \mu A_2), \\ g(\lambda, \mu) &= \det(B_0 - \lambda B_1 - \mu B_2) \end{aligned} \tag{2.1}$$

be two polynomials of order n and m, respectively; where $A_i \in R^{n\times n}$, $B_i \in R^{m\times m}$ satisfy the symmetric-definite condition stated in Section 1. We define eigencurves of $f(\lambda, \mu) = 0$ (or $g(\lambda, \mu) = 0$) as the curves in the λ–μ plane satisfying the equation $f = 0$ (or $g = 0$). Therefore, the problem is to find intersection points of two families of eigencurves. If $f(\lambda, \mu)$ and $g(\lambda, \mu)$ have no nonconstant common factors, then by Bezont's theorem [35, p. 44], the two families of eigencurves $f(\lambda, \mu) = 0$ and $g(\lambda, \mu) = 0$ meet at $n \times m$ points, counting their multiplicities. About the range of these intersection points, [24] gives a result similar to Gerschgorin's theorem.

The 2D bisection presented in [24] is based on continuity and monotonicity properties of the eigencurves, which are certainly true for the matrix triple (A_0, A_1, A_2) or (B_0, B_1, B_2) satisfying the "TBC" condition [24]. To extend the bisection method to full matrix cases, the key is to show the corresponding eigencurves still hold these properties. In [7, 8], Professors Binding and Browne investigated eigencurves of the differential equations (1.2). Under certain conditions, they pointed out, for any $n \geq 0$, the nth eigencurve $\mu^n(\lambda)$ of each equation in (1.2) is analytic over $\lambda \in R$ and satisfies

$$\mu^n(\lambda) = c\lambda + o(\lambda) \quad \text{as } \lambda \to \infty. \tag{2.2}$$

This tells us the asymptotic behavior of eigencurves of differential equations. For full matrices A_i, B_i $(i = 0, 1, 2)$ in the problem (1.1) with the symmetric-definite condition, the following properties can be easily verified.

LEMMA 1. *Any straight line $\lambda = \lambda_0/\mu = \mu_0$ has n intersection points (they may coincide) with eigencurves of $f(\lambda, \mu) = 0$, where $f(\lambda, \mu)$ is defined in* (2.1). *The n eigencurves $\mu^1(\lambda)$, ..., $\mu^n(\lambda)$ of $f(\lambda, \mu) = 0$ are continuous on $\{-\infty < \lambda < +\infty; -\infty < \mu < +\infty\}$ and each of them will cross once and only once any line $\lambda = \lambda_0/\mu = \mu_0$.*

The above lemma is also true for the m eigencurves of $g(\lambda, \mu) = 0$. Therefore, each of the eigencurves of $f(\lambda, \mu) = 0$ or $g(\lambda, \mu) = 0$ is strictly monotonic.

We are now in a position to describe an extension of the 2D bisection method. Given a rectangle $\Delta = [w_1, w_2; v_1, v_2]$, which is formed by four sides paralleling coordinate axes: $\lambda = w_1$, $\lambda = w_2$, $\mu = v_1$, $\mu = v_2$. Before considering the problem of how to locate intersection points (i.e., eigenpairs) of eigencurves $f(\lambda, \mu) = 0$ and $g(\lambda, \mu) = 0$ in this rectangle, we recall that if $M \in R^{n\times n}$ is a real symmetric matrix, then the leading principal minors of $M - \lambda I$ form a Sturm sequence $\{p_r(\lambda)\}_{r=0}^{n}$, where $p_0(\lambda) = 1$ by definition. Consequently the number of eigenvalues greater than λ is equal to the number of agreements in sign between consecutive members of the sequence $\{p_r(\lambda)\}_{r=0}^{n}$ (if some $p_r(\lambda)$ is zero then its sign is taken to be that of $p_{r-1}(\lambda)$). Furthermore, the matrix pencil $A - \lambda B$ possesses the same property if A is symmetric and B is symmetric positive definite; i.e., the number of eigenvalues of $Ax = \lambda Bx$ which are greater than λ is equal to the number of agreements in sign between consecutive members of the sequence $\{\det(A_r - \lambda B_r)\}_{r=0}^{n}$ with the definition of $\det(A_0 - \lambda B_0) \equiv 1$, where A_r, B_r are the leading principal submatrices of order r of A and B, respectively (see [33, 41]). Therefore, a bisection method associated with the Sturm sequence property can be employed to locate any individual eigenvalue of $Ax = \lambda Bx$.

For the problem (1.1), suppose A_i, B_i $(i = 0, 1, 2)$ satisfy the symmetric-definite condition. Hence, both families of eigencurves are monotonic and continuous on $\{-\infty < \lambda < +\infty; -\infty < \mu < +\infty\}$. Based on this observation, we can adopt two-dimensional bisection to locate the intersection points of $f(\lambda, \mu) = 0$ and $g(\lambda, \mu) = 0$ in the rectangle Δ. Note that if there are eigencurves of $f(\lambda, \mu) = 0$ (or $g(\lambda, \mu) = 0$) passing through the rectangle Δ, they must cross at least one side of Δ. On that side, however, either λ or μ is a constant. So the problem is turned into an ordinary generalized eigenvalue problem. By the Sturm sequence property, we can check if that side contains eigenvalues or not. If either eigencurves of $f(\lambda, \mu) = 0$ or that of $g(\lambda, \mu) = 0$ do not pass Δ, then there is no eigenpair inside Δ. Otherwise, divide Δ into four equal smaller rectangles and repeat the above process until both sides of the smaller rectangle are less than a required accuracy ε. Then the midpoint of the rectangle can be taken as an approximate eigenpair. If there is more than one eigencurve of $f(\lambda, \mu) = 0$ and/or $g(\lambda, \mu) = 0$ passing through the last rectangle, we consider this solution as the approximation of a group of roots (repeated or very closed roots).

Let $[w_1^{(j)}, w_2^{(j)}; v_1^{(j)}, v_2^{(j)}]$, $j = 0, 1, 2, \ldots$, be a sequence of rectangles obtained in the above 2D bisection method, where $[w_1^{(0)}, w_2^{(0)}; v_1^{(0)}, v_2^{(0)}] = [w_1, w_2; v_1, v_2]$. Notice that

$$|w_1^{(j)} - w_2^{(j)}| = 2^{-j}\,|w_1^{(0)} - w_2^{(0)}|,$$
$$|v_1^{(j)} - v_2^{(j)}| = 2^{-j}\,|v_1^{(0)} - v_2^{(0)}|. \quad (2.3)$$

In order to satisfy $|w_1^{(j)} - w_2^{(j)}| < \varepsilon$ and $|v_1^{(j)} - v_2^{(j)}| < \varepsilon$, the number of bisections has to meet the following inequality

$$j > \ln\left[\max\left(\frac{|w_1^{(0)} - w_2^{(0)}|}{\varepsilon}, \frac{|v_1^{(0)} - v_2^{(0)}|}{\varepsilon}\right)\right] \Big/ \ln 2. \quad (2.4)$$

The matrices A_i, B_i $(i = 0, 1, 2)$ in (2.1) are often tridiagonal in practice. In such cases, it is preferable to use the following sequence $\{s_i(\mu)\}_{i=1}^{n}$, say, for $\det(A_0 - \lambda A_1 - \mu A_2) = 0$ with $\lambda = \lambda_0$ ([33]; also cf. [42, pp. 249–250; 32, pp. 52–53]),

$$s_1(\mu) = a_{11}^{(0)} - \lambda_0 a_{11}^{(1)} - \mu a_{11}^{(2)}, \quad (2.5)$$
$$s_i(\mu) = a_{ii}^{(0)} - \lambda_0 a_{ii}^{(1)} - \mu a_{ii}^{(2)} - (a_{i,i-1}^{(0)} - \lambda_0 a_{i,i-1}^{(1)} - \mu a_{i,i-1}^{(2)})^2 / s_{i-1}(\mu). \quad (2.6)$$

where the $a_{ij}^{(l)}$ are entries of A_l $(i, j = 1, \ldots, n;\ l = 0, 1, 2)$, see the next section for more details.

When A_i, B_i $(i = 0, 1, 2)$ are large, band matrices and the band-widths are much smaller than the orders of A_i and B_i, it is important to take advantage of the band structure. We may use a modification of elimination method to determine the sign of $\{\det(A_{0,r} - \lambda A_{1,r} - \mu A_{2,r})\}_{r=0}^{n}$ and $\{\det(B_{0,r} - \lambda B_{1,r} - \mu B_{2,r})\}_{r=0}^{m}$, in which the reduction process involved in triangular decomposition with pivoting is applied to only the first $r + 1$ rows, for successive values of r from 1 to $n - 1$ (cf. [20] for details).

Once an eigenpair has been isolated, the method of successive linear interpolation can be used to obtain a higher convergence rate (cf. [33]).

3. ROUNDING ERROR ANALYSIS

It is well known that the ordinary bisection method is very stable [41]. In this section, we will examine the effect of rounding-error arising from the 2D bisection process for the problem (1.1). For simplicity, we suppose A_i, B_i $(i = 0, 1, 2)$ are all symmetric tridiagonal.

Consider a typical step of the algorithm. Fix $\lambda := \lambda_0$ in the first equation of (1.1) and compute the sequence $\varphi_1(\lambda_0, \mu)$, $\varphi_2(\lambda_0, \mu)$, ..., $\varphi_n(\lambda_0, \mu)$ according to the formulae

$$\varphi_1(\lambda_0, \mu) = a_{11}^{(0)} - \lambda_0 a_{11}^{(1)} - \mu a_{11}^{(2)}, \quad (3.1)$$
$$\varphi_i(\lambda_0, \mu) = (a_{ii}^{(0)} - \lambda_0 a_{ii}^{(1)} - \mu a_{ii}^{(2)})\varphi_{i-1}(\lambda_0, \mu) - (a_{i,i-1}^{(0)} - \lambda_0 a_{i,i-1}^{(1)} - \mu a_{i,i-1}^{(2)})^2 \varphi_{i-2}(\lambda_0, \mu) \quad (3.2)$$

$(i = 2, 3, \ldots, n)$, where $a_{ij}^{(l)}$ is the entry (i, j) of A_l $(i, j = 1, 2, \ldots, n;\ l = 0, 1, 2)$ and $\varphi_0(\lambda_0, \mu) \equiv 1$ by definition.

In order to prevent overflow and underflow, the following sequence $\{s_i(\lambda_0, \mu)\}_{i=1}^{n}$ is actually calculated

$$s_1(\lambda_0, \mu) = \frac{\varphi_1(\lambda_0, \mu)}{\varphi_0(\lambda_0, \mu)} = a_{11}^{(0)} - \lambda_0 a_{11}^{(1)} - \mu a_{11}^{(2)}, \tag{3.3}$$

$$\begin{aligned} s_i(\lambda_0, \mu) &= \frac{\varphi_i(\lambda_0, \mu)}{\varphi_{i-1}(\lambda_0, \mu)} = [(a_{ii}^{(0)} - \lambda_0 a_{ii}^{(1)} - \mu a_{ii}^{(2)})\varphi_{i-1}(\lambda_0, \mu) \\ &\quad - (a_{i,i-1}^{(0)} - \lambda_0 a_{i,i-1}^{(1)} \\ &\quad - \mu a_{i,i-1}^{(2)})^2 \varphi_{i-2}(\lambda_0, \mu)]/\varphi_{i-1}(\lambda_0, \mu) \\ &= \begin{cases} a_{ii}^{(0)} - \lambda_0 a_{ii}^{(1)} - \mu a_{ii}^{(2)} \\ \quad - \dfrac{(a_{i,i-1}^{(0)} - \lambda_0 a_{i,i-1}^{(1)} - \mu a_{i,i-1}^{(2)})^2}{s_{i-1}(\lambda_0\ \mu)} \\ \qquad\qquad (\text{when } s_{i-1}s_{i-2} \neq 0) \\ a_{ii}^{(0)} - \lambda_0 a_{ii}^{(1)} - \mu a_{ii}^{(2)} \\ \qquad\qquad (\text{when } s_{i-1} \neq 0, \text{ but } s_{i-2} = 0) \\ -\infty \\ \quad (\text{when } s_{i-2} \neq 0, \text{ but } s_{i-1} = 0), i = 2, 3, ..., n. \end{cases} \end{aligned} \tag{3.4}$$

Then the Sturm sequence count is given by the number of positive $s_i(\mu)$, $i = 1, ..., n$ (cf. [33]).

By the backward error analysis [40], we can prove the following theorem 1. Let $\mathrm{fl}(\cdot)$ stand for the result by performing the appropriate floating-point operation $(\cdot)$. Let $\circ$ denote one of the four operations $+, -, \times, \div$. Because of rounding errors, we have

$$\mathrm{fl}(x \circ y) = (x \circ y)(1 + \delta), \quad |\delta| \leq 2^{-t}. \tag{3.5}$$

where x and y are standard floating-point numbers, t (here and below) is the number of digits after the binary point for the computer in use. If $|\delta_i| \leq 2^{-t}$ $(i = 1, 2, ..., n)$ and $n \cdot 2^{-t} \leq 0.01$, then, when $n > 2$, we have (e.g., cf. [17, pp. 92–93])

$$1 - n \cdot 2^{-t} \leq \prod_{i=1}^{n} (1 + \delta_i) \leq 1 + 1.01 \cdot n \cdot 2^{-t}. \tag{3.6}$$

THEOREM 1. *For any value of* μ *the computed values of the sequence* $\{s_i(\lambda_0, \mu)\}_{i=1}^n$ *are the exact values corresponding to the modified tridiagonal matrices* $A_0 + \delta A_0$, A_1, $+ \delta A_1$, *and* $A_2 + \delta A_2$. *Let* $\delta a_{ij}^{(l)}$ *denote the entry* (i, j) *of* δA_l $(i, j = 1, 2, ..., n;\ l = 0, 1, 2)$; *then*

$$\begin{aligned} |\delta a_{ii}^{(l)}| &\leq (3.03)2^{-t}|a_{ii}^{(l)}|, \quad l = 0, 2; \\ |\delta a_{i,i-1}^{(l)}| &\leq (3.54)2^{-t}|a_{i,i-1}^{(l)}|, \quad l = 0, 2; \\ |\delta a_{ii}^{(1)}| &\leq (4.04)2^{-t}|a_{ii}^{(1)}|; \\ |\delta a_{i,i-1}^{(1)}| &\leq (4.55)2^{-t}|a_{i,i-1}^{(1)}|. \end{aligned} \tag{3.7}$$

Proof. The proof is by induction for the index m: $1 \leq m \leq n$.

It is not difficult to verify that Theorem 1 is true for $m = 1, 2$. Let us assume that (3.7) hold for $m \leq k$, i.e., the computed $s_1(\lambda_0, \mu), ..., s_k(\lambda_0, \mu)$ are exact for three matrices having modified elements up to $a_{kk}^{(l)} + \delta a_{kk}^{(l)}$, $a_{k,k-1}^{(l)} + \delta a_{k,k-1}^{(l)}$ $(l = 0, 1, 2)$ and the modifications satisfy (3.7), then we show that the computed $s_{k+1}(\lambda_0, \mu)$ is the exact value for three matrices having these same modified elements in rows 1, 2, ..., k and modified elements $a_{k+1,k+1}^{(l)} + \delta a_{k+1,k+1}^{(l)}$, $a_{k+1,k}^{(l)} + \delta a_{k+1,k}^{(l)}$ $(l = 0, 1, 2)$.

Suppose $s_k(\lambda_0, \mu)s_{k-1}(\lambda_0, \mu) \neq 0$,[1]

$$\begin{aligned} &s_{k+1}(\lambda_0, \mu) \\ &= \mathrm{fl}\left(a_{k+1,k+1}^{(0)} - \lambda_0 a_{k+1,k+1}^{(1)} - \mu a_{k+1,k+1}^{(2)} - \frac{(a_{k+1,k}^{(0)} - \lambda_0 a_{k+1,k}^{(1)} - \mu a_{k+1,k}^{(2)})^2}{s_k(\lambda_0, \mu)}\right) \\ &= \mathrm{fl}(a_{k+1,k+1}^{(0)} - \lambda_0 a_{k+1,k+1}^{(1)} - \mu a_{k+1,k+1}^{(2)})(1 + \delta_1) \\ &\quad - \frac{(\mathrm{fl}(a_{k+1,k}^{(0)} - \lambda_0 a_{k+1,k}^{(1)} - \mu a_{k+1,k}^{(2)})(1 + \delta_2))^2}{s_k(\lambda_0, \mu)} \\ &= a_{k+1,k+1}^{(0)}(1 + \varepsilon_0') - \lambda_0 a_{k+1,k+1}^{(1)}(1 + \varepsilon_1') - \mu a_{k+1,k+1}^{(2)}(1 + \varepsilon_2') \\ &\quad - \frac{[a_{k+1,k}^{(0)}(1 + \delta_0') - \lambda_0 a_{k+1,k}^{(1)}(1 + \delta_1') - \mu a_{k+1,k}^{(2)}(1 + \delta_2')]^2}{s_1(\lambda_0, \mu)}. \end{aligned}$$

Where $|\delta_1| \leq 2^{-t}$, $(1 - 2^{-t})^{3/2} \leq 1 + \delta_2 \leq (1 + 2^{-t})^{3/2}$. So

$$\begin{aligned} (1 - 2^{-t})^3 &\leq (1 + \varepsilon_l') \leq (1 + 2^{-t})^3, \quad l = 0, 2; \\ (1 - 2^{-t})^4 &\leq (1 + \varepsilon_1') \leq (1 + 2^{-t})^4; \\ (1 - 2^{-t})^{7/2} &\leq (1 + \delta_l') \leq (1 + 2^{-t})^{7/2}, \quad l = 0, 2; \\ (1 - 2^{-t})^{9/2} &\leq (1 + \delta_1') \leq (1 + 2^{-t})^{9/2}. \end{aligned} \tag{3.8}$$

Therefore,

$$\begin{aligned} |\delta a_{k+1,k+1}^{(l)}| &= |a_{k+1,k+1}^{(l)}\varepsilon_l'| \leq (3.03)2^{-t}|a_{k+1,k+1}^{(l)}|, \quad l = 0, 2; \\ |\delta a_{k+1,k}^{(l)}| &= |a_{k+1,k}^{(l)}\delta_l'| \leq (3.54)2^{-t}|a_{k+1,k}^{(l)}|, \quad l = 0, 2; \\ |\delta a_{k+1,k+1}^{(1)}| &= |a_{k+1,k+1}^{(1)}\varepsilon_1'| \leq (4.04)2^{-t}|a_{k+1,k+1}^{(1)}|; \\ |\delta a_{k+1,k}^{(1)}| &= |a_{k+1,k}^{(1)}\delta_1'| \leq (4.55)2^{-t}|a_{k+1,k}^{(1)}|. \end{aligned} \tag{3.9}$$

The proof is then completed. ∎

[1] If $s_k(\lambda_0, \mu) = 0$, the conclusion is evident; if $s_{k-1}(\lambda_0, \mu) = 0$, there are no operations for $a_{k+1,k}^{l}$ $(l = 0, 1, 2)$; hence the proof can be simplified.

As the theorem shows, the 2D bisection method is very stable for the tridiagonal matrix triples satisfying the symmetric-definite condition.

4. DEFINITENESS CONDITION AND DISCRETIZATION ERROR ESTIMATE

As we stated in the Introduction, the problem (1.1) arises from the two-parameter S-L eigenvalue problem (1.2). Therefore it is needed to estimate the discretization error when (1.2) is replaced by (1.1). To this end, we first introduce an important concept—the *definiteness condition.* We can describe this condition for more general multiparameter eigenvalue problems as

$$\begin{gathered} \lambda_0 M_{10} u_1 + \lambda_1 M_{11} u_1 + \cdots + \lambda_k M_{1k} u_1 = 0, \\ \vdots \\ \lambda_0 M_{k0} u_k + \lambda_1 M_{k1} u_k + \cdots + \lambda_k M_{kk} u_k = 0, \end{gathered} \tag{4.1}$$

where $u_i \in C^{n_i}$ $(i = 1, ..., k)$, $M_{ij} \in C^{n_i \times n_i}$ $(i = 1, ..., k; j = 0, 1, ..., k)$ are Hermitian matrices over the complex field. A nonzero $(k + 1)$-tuple of scalars $\lambda = (\lambda_0, ..., \lambda_k)$ is called an eigenvalue such that there exist vectors u_r, $r = 1, ..., k$, satisfying the k equations in (4.1). The corresponding tensor product $u = u_1 \otimes \cdots \otimes u_k$ is called the eigenvector of the problem (4.1). We introduce the following.

Definiteness Condition. For some fixed set of real scalars $\mu_0, ..., \mu_k$, and for all sets

$$g_i \in C^{n_i}, \quad g_i \neq 0, \quad i = 1, ..., k, \tag{4.2}$$

we have

$$\det \begin{pmatrix} \mu_0 & \cdots & \mu_k \\ g_1^H M_{10} g_1 & \cdots & g_1^H M_{1k} g_1 \\ & \vdots & \\ g_k^H M_{k0} g_k & \cdots & g_k^H M_{kk} g_k \end{pmatrix} > 0. \tag{4.3}$$

Atkinson [3] proved that if the above definiteness condition holds, then all eigenvalues are real and the number of distinct eigenvalues does not exceed $\prod_{r=1}^{k} n_r$; there is a complete set of eigenvectors, orthogonal in certain sense, where to each eigenvalue is associated a number of eigenvectors equal to its multiplicity.

Now we turn to the discretization error of the problem (1.1) from the problem (1.2). We will treat the case of two parameters; however, the results can be easily extended to multiparameter cases. Suppose that $\{x_i^{(1)}\}_{i=1}^{n}$ and $\{x_i^{(2)}\}_{i=1}^{m}$ are two nets of grid points on $[a_1, b_1]$ and $[a_2, b_2]$, respectively, and that some discrete technique is employed to the equations in (1.2), then the discrete equations

$$\begin{aligned} (A_0 - \lambda A_1 - \mu A_2) z_1 &= \tau_1(z_1), \\ (B_0 - \lambda B_1 - \mu B_2) z_2 &= \tau_2(z_2), \end{aligned} \tag{4.4}$$

are formed, where $A_i \in R^{n\times n}$, $B_i \in R^{m\times m}$ $(i = 0, 1, 2)$, $z_1 = [y_1(x_1^{(1)}), ..., y_1(x_n^{(1)})]^{\mathrm{T}}$, $z_2 = [y_2(x_1^{(2)}), ..., y_2(x_m^{(2)})]^{\mathrm{T}}$, $\tau_1(z_1)$ and $\tau_2(z_2)$ are local truncation errors. In practical computation, the matrix equations

$$\begin{aligned} (A_0 - \tilde{\lambda} A_1 - \tilde{\mu} A_2) \tilde{z}_1 &= 0, \\ (B_0 - \tilde{\lambda} B_1 - \tilde{\mu} B_2) \tilde{z}_2 &= 0, \end{aligned} \tag{4.5}$$

are solved, where $(\tilde{\lambda}, \tilde{\mu})$ is intended to be an approximation to some eigenpair (λ, μ) of (4.4) and the net function $\tilde{z}_1 \otimes \tilde{z}_2$ is intended to be an approximation to the corresponding eigenfunction on the net. The error in eigenpairs is denoted by $e = (\lambda, \mu)^{\mathrm{T}} - (\tilde{\lambda}, \tilde{\mu})^{\mathrm{T}}$.

In [22] the author gave the following error estimation.

THEOREM 2. *Suppose that* $\tilde{z}_1 = z_1 + o(1)$, $\tilde{z}_2 = z_2 + o(1)$, *and* $\|z_1\| = \|z_2\| = 1$. *If*

$$\left| \det \begin{pmatrix} z_1^T A_1 z_1 & z_1^T A_2 z_1 \\ z_2^T B_1 z_2 & z_2^T B_2 z_2 \end{pmatrix} \right| \geq \delta > 0, \tag{4.6}$$

then

$$\|e\|_2 \leq c \cdot \delta^{-1} (\|\tau_1\|_2^2 + \|\tau_2\|_2^2)^{1/2}, \tag{4.7}$$

where c is a constant independent of the truncation errors.

Notice that condition (4.6) is somewhat stronger than (4.3) in the sense that we let $\mu_0 \neq 0$, $\mu_1 = \mu_2 = 0$.

The estimation (4.7) also holds if

$$\left| \det \begin{pmatrix} \tilde{z}_1^T A_1 \tilde{z}_1 & \tilde{z}_1^T A_2 \tilde{z}_1 \\ \tilde{z}_2^T B_1 \tilde{z}_2 & \tilde{z}_2^T B_2 \tilde{z}_2 \end{pmatrix} \right| \geq \delta > 0. \tag{4.8}$$

In application, the matrices A_i, B_i are usually symmetric. However, in derivation of the above results, we did not exploit the symmetricity of A_i, B_i; instead, we invoke the conditions $\tilde{z}_1 = z_1 + o(1)$, $\tilde{z}_2 = z_2 + o(1)$. These conditions are not necessary if the following condition (4.9) holds.

THEOREM 3. *Suppose A_i, B_i ($i = 0, 1, 2$) are real and symmetric. If the net functions z_1, z_2 and their approximations $\tilde{z}_1$, $\tilde{z}_2$ satisfy $\|z_1\| = \|z_2\| = \|\tilde{z}_1\| = \|\tilde{z}_2\| = 1$ and*

$$\left|\det\begin{pmatrix} z_1^T A_1 \tilde{z}_1 & z_1^T A_2 \tilde{z}_1 \\ z_2^T B_1 \tilde{z}_2 & z_2^T B_2 \tilde{z}_2 \end{pmatrix}\right| \geq \delta > 0, \tag{4.9}$$

then

$$\|e\|_1 \leq c \cdot \delta^{-1}(\|\tau_1\|_2 + \|\tau_2\|_2), \tag{4.10}$$

$$\|e\|_2 \leq c \cdot \delta^{-1}(\|\tau_1\|_2^2 + \|\tau_2\|_2^2)^{1/2}, \tag{4.11}$$

$$\|e\|_\infty \leq c \cdot \delta^{-1} \max\{\|\tau_1\|_2, \|\tau_2\|_2\}, \tag{4.12}$$

where c is a constant independent of the truncation errors.

Proof. From the first equations of (4.4) and (4.5), we have

$$\tilde{z}_1^T A_0 z_1 - \lambda \tilde{z}_1^T A_1 z_1 - \mu \tilde{z}_1^T A_2 z_1 = \tilde{z}_1^T \tau_1(z_1), \tag{4.13}$$

$$z_1^T A_0 \tilde{z}_1 - \tilde{\lambda} z_1^T A_1 \tilde{z}_1 - \tilde{\mu} z_1^T A_2 \tilde{z}_1 = 0. \tag{4.14}$$

Note that A_i ($i = 0, 1, 2$) are symmetric, and subtracting (4.13) from (4.14) gives

$$(\lambda - \tilde{\lambda}) z_1^T A_1 \tilde{z}_1 + (\mu - \tilde{\mu}) z_1^T A_2 \tilde{z}_1 = -\tilde{z}_1^T \tau_1(z_1). \tag{4.15}$$

Similarly, we have

$$(\lambda - \tilde{\lambda}) z_2^T B_1 \tilde{z}_2 + (\mu - \tilde{\mu}) z_2^T B_2 \tilde{z}_2 = -\tilde{z}_2^T \tau_2(z_2). \tag{4.16}$$

Combining (4.15) and (4.16) gives

$$e = \begin{pmatrix} \lambda - \tilde{\lambda} \\ \mu - \tilde{\mu} \end{pmatrix} = -\begin{pmatrix} z_1^T A_1 \tilde{z}_1 & z_1^T A_2 \tilde{z}_1 \\ z_2^T B_1 \tilde{z}_2 & z_2^T B_2 \tilde{z}_2 \end{pmatrix}^{-1} \begin{pmatrix} \tilde{z}_1^T \tau_1(z_1) \\ \tilde{z}_2^T \tau_2(z_2) \end{pmatrix}.$$

Hence

$$\|e\|_p \leq c \cdot \delta^{-1} \left\| \begin{pmatrix} \tilde{z}_1^T \tau_1(z_1) \\ \tilde{z}_2^T \tau_2(z_2) \end{pmatrix} \right\|_p, \tag{4.17}$$

where $p = 1, 2, \infty$ (cf. [19]). The conclusions of the theorem are then evident. ∎

It is easy to see that the truncation error estimate is closely related to the perturbation analysis of the eigenvalues. Although the perturbation theory of ordinary matrix eigenvalue problems has a long history and is quite developed (e.g., cf. [37]), whether or not those techniques and skills can be applied to multiparameter eigenvalue problems is an open question, as one might notice that both Theorems 2 and 3 are not ready to be applied in practice. Further work is necessary, and this virgin field is waiting for us to exploit.

5. NUMERICAL EXAMPLES

In this section we present some numerical results. The accuracy of the results is limited by two factors. One is the truncation error. As shown in Section 4, the exact solution of the discrete double eigenvalue problem (1.1) is only an approximate solution to the original differential problem (1.2). The other is roundoff error. In most cases, it is impossible to get the exact solution of (1.1) due to finite-digit operation. It is not appropriate to set the termination criterion of the 2D bisection less than (in order) the truncation error.

The first example in this section is a model problem of which we know the exact solution. A comparison of computed solutions with the 2D bisection presented in [24] has been made in terms of CPU time and accuracy. The result illustrates the necessity of our extension. The second example was considered by Fox, Blum, *et al.* The last example, which has applications in electrostatic fields, is taken from [28]. All computations were carried out in 32-bit arithmetic on a CYBER 860 computer.

EXAMPLE 1. Consider a model double eigenvalue problem

$$\begin{aligned} y_1'' + 2\lambda y_1 + \mu y_1 &= 0, \\ y_2'' + \lambda y_2 + 2\mu y_2 &= 0, \\ y_1(0) = y_1(\pi) &= 0, \\ y_2(0) = y_2(\pi) &= 0. \end{aligned} \tag{5.1}$$

The exact eigenpairs are $(\lambda, \mu) = ((2(k+1)^2 - (l+1)^2)/3, (2(l+1)^2 - (k+1)^2)/3)$, $k, l = 0, 1, 2, \ldots$.

By Numerov's method with uniform mesh length $h = \pi/(n+1)$, the approximative matrix equations were formed,

$$\begin{aligned} A_0 x + \lambda A_1 x + \mu A_2 x &= 0, \\ B_0 y + \lambda B_1 y + \mu B_2 y &= 0, \end{aligned} \tag{5.2}$$

where $x, y \in R^n$, $A_0 = B_0 = (c_{ij})$ are n by n symmetric tridiagonal matrices with $c_{ii} := -2$ ($i = 1, \ldots, n$), $c_{i,i+1} := 1$ ($i = 1, \ldots, n-1$), $A_1 = B_2 = h^2(2I + \frac{1}{6}A_0)$, $A_2 = B_1 = h^2(I + \frac{1}{12}A_0)$, I is identity matrix. Note that the 2D bisection in [24] cannot be applied in this case.

Since A_1, A_2, B_1, and B_2 are all positive definite, we use the algorithm described in Section 2 to solve eigenpairs in the rectangle $[-10, 17; -10, 17]$. Taking $n = 199$, the results coincide with the exact solutions to at least five decimal places. The CPU time was 54 s.

If we adopt the center difference scheme to approximate (5.1), A_1, A_2, B_1, and B_2 in (5.2) will be diagonal and definite. Therefore we can use the 2D bisection method in [24]. By taking $n = 499$ with uniform mesh, the computed results coincide with the exact solutions to two decimal places within a CPU time of 142 s.

EXAMPLE 2. [10, 18] studied the example

$$\begin{aligned} &y'' + \frac{1}{p(x)}(\lambda + \mu x + x^2)y = 0, \\ &y(-1) = y(0) = y(1), \end{aligned} \tag{5.3}$$

where $p(x) = 1 + x + x^2$.

To discrete (5.3), take uniform mesh points $x_1^{(1)}, \ldots, x_n^{(1)}$ in $(-1, 0)$ and $x_1^{(2)}, \ldots, x_n^{(2)}$ in $(0, 1)$, so that the step size is $h = 1/(n + 1)$. Since $y(0) = 0$, $y'(0) \neq 0$, we assume $y(x_n^{(1)}) \neq 0$, $y(x_1^{(2)}) \neq 0$ and let the eigenvectors satisfy $(y(x_1^{(2)}) - y(0))/h = -(y(x_n^{(1)}) - y(0))/h$. Using the algorithm described in Section 2, the eigenpairs in the rectangle [0, 200; 0, 200] are solved as

$$\begin{aligned} &(12.13506, \quad 9.60367), \\ &(36.60801, \quad 68.56718), \\ &(50.51166, \quad 36.33246), \\ &(71.47327, \quad 182.52810), \\ &(95.77223, \quad 124.53363), \\ &(114.55835, \quad 80.77256), \\ &(140.76062, \quad 21.29578), \end{aligned}$$

in which one pair of eignevalues $\lambda = 12.135$, $\mu = 9.604$ was given in [18], and $\lambda = 12.134545$, $\mu = 9.6032854$ in [10].

EXAMPLE 3. Finally we solve the charge-singularity problem considered in [28]; [5] rewrites it in the self-adjoint form

$$\begin{aligned} &((1 - k^2\cos^2 x)^{1/2}L')' + (\lambda_1 + \lambda_2 k^2 \sin^2 x) \\ &\qquad \times (1 - k^2\cos^2 x)^{-1/2}L = 0, \\ &((1 - k'^2\cos^2 y)^{1/2}N')' + (-\lambda_1 + \lambda_2 k'^2 \sin^2 y) \\ &\qquad \times (1 - k'^2\cos^2 y)^{-1/2}N = 0, \\ &x, y \in (0, x), \\ &L(0) = L'(\pi) = 0, \\ &N'(0) = N'(\pi) = 0 \quad \text{if } 0 < \chi < \pi, \\ &N(0) = N(\pi) = 0 \quad \text{if } \pi < \chi < 2\pi, \end{aligned} \tag{5.4}$$

where $k = \sin(1/2|\pi - \chi|)$, $k' = \sqrt{1 - k^2}$, and χ is the angle of the sector. The problem is divided into three cases: spike when χ is small, slit when χ is close to 2π, and nearly straight edge when χ is close to π. We use Marčuk's integral identity to discrete the differential equations in (5.4) (cf. [4, Chap. 4]). The computed results are given in Table I.

TABLE I

χ/π	λ_2 (Morrison)	λ_2 (Bailey)	λ_2 (computed)
0.04021	0.134437	0.1344183	0.1344238
0.28858	0.275451	0.2754502	0.2753945
0.9500	0.701815	[a]	0.7025342
1.125	0.887700	0.8877101	0.8876282
1.875	1.971545	[a]	1.971277
1.950	1.995385	[a]	1.995327

[a] Results not given in [5].

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